MANAGEMENT SCIENCE
Vol. 20, No. 3, November, 1973
Printed in U.S.A.

PROFIT ANALYSES AND SEQUENTIAL BID PRICING MODELS*

K. O. KORTANEK§, J. V. SODEN¶ AND D. SODARO‡†

with a statement by G. R. Seiler

A bid pricing strategy based upon opportunity costs is presented for the firm confronted with the problem of competitively pricing a sequence of sealed tenders for future undifferentiated but interrelated contract work. Each contract, if awarded, will require the expending of predetermined amounts of several restricted resources at a later time. A goal of the bidding strategy is to determine a price structure of the winning bids which maximizes the total contribution over direct costs associated with the time period of resource utilization. According to various levels of complexity of data, several models of the problem are developed together with optimal bid price rules. Each optimal rule involves scarce resource "cost" charges for future opportunities and competitive advantages according to the general bid rule form: OPTIMAL BID PRICE = DIRECT COSTS + OPPORTUNITY COSTS + COMPETITIVE ADVANTAGE FEE.

Experiences are also given of an adaptive and conditional implementation of the general bid rule for a major chemical manufacturer in a complex business area where sales orders are determined by competitive bidding.

1. Introduction

In this paper we explore the problem confronting a firm that will be bidding sequentially via sealed tenders and against several competitors for contracts whose fulfillment (or production) will occur during a later fixed time interval. Each contract, if won, will require the use of predetermined amounts or restricted resources at the time of actual production. The firm must determine a bidding strategy that will optimize its total awarded contract value, less direct costs to be incurred, subject to possibly flexible resource constraints. "Flexible" means that the actual resource commitment at the close of bidding is such that short-term alterations are possible. A successful bidding strategy must also focus on the possibility of technological alterations or improvements, as a consequence of the market moving toward an equilibrium among competitors.

The notion of equilibrium among competitors is important to the bidding environment we study. We have addressed a complex multiple contract competitive bidding process where (1) no one competitor is uniformly better than everyone else, (2) the "efficiency" of a given competitor depends on the relationship between contracts he receives including their many joint production possibilities. There may be no "lowest cost" competitor in the collection of players. Rather, a natural kind of equilibrium among all players is sought. Such an equilibrium among players depends on varying degrees of individual player technological and managerial capabilities, which as far as other competitors are concerned are essentially unknown to any given player.

* Received July 1970 by Professor Arthur M. Geoffrion, Departmental Editor, *Integer and Large-Scale Programming*; revised April 1972.

§ Carnegie-Mellon University.
¶ McKinsey & Company, Inc.
‡ Environmental Leisure Corporation.

† This report is a revision of the earlier Technical Report No. 66, Department of Operations Research, Cornell University, March 1969, supported by the National Science Foundation Grants GK-1581, GP-7798, at Cornell University. The authors acknowledge helpful written comments and criticisms by Dr. S. C. Littlechild.

Various degrees of uncertainty and complexity exist for the data of the bidding environment. Most data are probabilistic and even conditional on past data and past decisions made. The complexities of a realistic environment have led us to organize the paper according to various "levels of difficulty" for mathematical models of the bidding process. By a "level of difficulty" we mean a description of complexity in the sequential bidding process. The description is given in terms of knowledge about: market price of a contract, order size, probability of winning, technological production coefficients, conditional interdependencies of data for bidding, and which contracts are to appear for bidding and when are they to appear.

We have chosen a simple hierarchy of four possible levels of difficulty. Mathematical models are constructed for each level according to degrees of complexity in the data. Then, an optimal bid price rule is determined for each level of complexity. Each of these rules translates verbally into the same bid price policy for all levels of difficulty. The various levels are described as follows.

Level of Difficulty (α). Complete knowledge is available on all contracts, market prices, estimated annual volume, and when bids are to be placed and for which contracts. Level (α) is also called the "first level" and is described in §3. The first level of difficulty also provides guidelines with which any bidding procedure may be evaluated. Examples of ad hoc bidding procedures are given in §3.1. In §§3.2 and 3.3 optimal bid rules are determined for level (α) by using elementary linear programming theory.

The next two levels are developed in Appendix 1. They give insight into an optimal form of a bid price rule under more realistic conditions.

Level of Difficulty (β). Market price p_j of contract j is a random variable while the response function giving the probability of winning contract j at bid price b_j is known. When the goal is to maximize expected markup, Theorems I and II of Appendix 1 give optimal bid price formulae. Theorem III gives information on the optimal bid price when one seeks to maximize the probability of attaining a prespecified level of absolute total markup.

Level of Difficulty (γ). The market price is also random, but the probability of winning is now conditional on the bid price b_j and a general market price level. Technological production coefficients are also chance variables as well as unit direct costs. Theorems IV and V of Appendix 1 derive optimal bid price rules under these assumptions.

Level of Difficulty (δ). Here, much less is known about which contracts are to appear and when they are to appear than in levels (α) through (γ). The measurement of the opportunity cost of scarce resources becomes more difficult. In §4 of the text this level is discussed. Based on the concept of utilization of capacity and our findings at previous levels, we suggest a bid price policy rule of the form encountered at all the other levels. The theory and concept of the generalized restricted resource use approach to a "real world" bidding level of difficulty "(δ)" is discussed in §2. §2 also contains some management responses to the judgment required for implementation of a mathematical model related to developments made in levels (α)—(γ) as part of a total bidding system.

From the viewpoint of a given bidder, little is assumed to be known about the other bidders' technology and methods of bidding. Therefore measurements of their behavior must be based on market reaction to submitted bid prices over a period of time. Another characteristic of the process is that only a small fraction of the winning prices of all the bids made by all competitors is known to any given competitor. Each knows however (at any stage of bidding) the total dollar amount made available to all competitors for bidding. Thus, he is able to estimate his own market penetration at any stage of bidding.

The characteristics of this particular bidding environment touch upon concepts of most realistic environments. For example: How do short-run operations relate to longer term ones? When is it wise to listen to the market and initiate substantial technological change? How can one assure that, at the end of the bidding period, total overhead will be absorbed? Accountants and economists have done little on the overhead absorption problem in sequential bidding processes and, in general, warn against allocating overhead charges on a per unit basis. The corporate manager is perplexed about these recommendations because he needs some method of incorporating the charges into his pricing mechanism so as not to distort an optimal pattern of wins and yet give reasonable assurance of overhead coverage at the close of bidding. The models derived in this paper give insight into these and other policy questions.

2. Profit Analyses and Competitive Bidding: Concept and Theory by George R. Seiler

A major chemical manufacturer has undertaken extensive analyses to maximize profits from producing units being run at full capacity. The objective is to obtain the largest profit contribution possible for each unit of time associated with the full operation. Our approach is to identify the limiting resource or resources and those items of cost which are truly variable with respect to time. We then work toward maximizing the revenue contribution less these variable or direct costs per unit of time.

For example, in one of the operations today a given production unit is being used to capacity. This unit may be used to produce a number of different products, and these products have a wide range of characteristics as to composition, production rate and sales price. In this type of machine operation, all costs other than raw materials, packaging supplies and freight out are essentially fixed over a moderate period of time. Based on the concept that everything other than the materials for the product are time charges, we can maximize the profit contribution over a period of time by maximizing the absolute contribution over direct cost (called "C.O.D."). Thus if we know the net unit selling price and the net unit materials cost plus the number of units that can be produced in a time period such as an hour, we can rank each of our products in terms of dollar C.O.D. per machine hour. This ranking is independent of overhead expenses, does not deal with sales margins, and is far removed from traditional accounting approaches. It is an analytical tool which operates to maximize the net dollars received per unit of time. If, for example, we can move three times the volume of a $1.00 C.O.D. item in the same time period as we can move one volume of a $2.50 C.O.D. item, we will select the $1.00 item.

We first applied this tool in detail, and with success, to a multiple-machine production process involving automotive products. We have recently made an extensive study of the other operations on this basis, and have greatly modified our product mix targets. We are also using this approach in simpler situations where only one machine is our limiting resource. The ideal approach is to run those items in descending rank order until one runs out of capacity. In addition, we are putting our marketing emphasis on those items which yield the maximum C.O.D. per unit of time.

This same generalized approach can also be applied where there is present overcapacity. In this case, production of items with positive C.O.D. is better than allowing idle time on the equipment, again based on the concept of a fixed overhead. A C.O.D. analysis can be made and incremental business programs can be put into effect to improve profitability. Concurrent programs are under way which review the overhead time charges associated with our operations and analyze them in terms of time rather than cost per unit. Any decrease in these charges over any time period increases the profitability for that time period given the same direct costs and sales revenue relationships.

There are varied levels of sophistication for the C.O.D. analysis. We have initiated recent work in a simple manual form so that it is easily understood by all users. We envision places, similar to multiple-machine processes, where more sophisticated modeling techniques may be used to assist us in long-term marketing strategies, scheduling operations, and decision making in complex situations. Standard cost information now being developed will serve as a data bank for this type of analysis.

In the multiple-machine process instance mentioned above, our system also includes a sophisticated computerized mathematical model to aid in dealing with the complex business of competitive bidding. Our bidding system consists of requests for quotations from the customer, establishment of standard production costs (direct costs and machine use rates), use of the mathematical model, interpretation of the model results, the submission of bids, and the feedback of information from end results. The mathematical model is, therefore, only a portion of our total system although it is one of the key elements. However, any evaluation of the system must be of the system as a whole. Therefore, we will review the concept, the model and the inputs and use of model results.

The concept, of course, is that of maximizing C.O.D. per unit of time as just outlined. The model was developed to determine best economic solutions within stipulated limitations and assessing the economic costs of the restrictions and, therefore, the value of altering them. Our model employs linear programming with dual evaluators to determine pricing that will tend to maximize C.O.D. We use the dual evaluators because they best express our realistic constraints: machine resources and established market limits for volume and types of contracts.

We have two major dual evaluators which are used in preparing bids: The first is *machine resources* and is expressed in terms of machine hour rates (determined by opportunity cost calculations) and setup charges. The setup charges are a predetermined function based on an average of 12 runs of each part made per year (which was based on customer release patterns); the setup is in hours and relates back to machine capacities and the determined machine hour rates. The second dual evaluator of major importance has been called the "*Competitive Advantage Fee*" ("CAF"). This is a representation of how well the part fits our remaining resources, or, in other words, how good it is to make with what we have left. The limiting factor here is the market reality rather than the production constraints. With this evaluator we attempt to measure our expectation of our achievable share of the total market for the bidding year.

The input to our model for producing any one part involves a number of items. First are the standards in terms of direct labor, raw materials and efficiencies. Second is the standard running times on each of our machines.

We began each year with a target C.O.D. based on a number of assumptions regarding our expectations for the full year. Throughout the bidding period we shift the dual evaluators (set new limits) based on performance to date. This is done on a periodic basis.

The bid pricing results generated by the model may be tempered based on market intelligence and relative achievement on past bids. This, in turn, becomes a part of the input to our periodic shifting of dual evaluators. In addition, certain contingency plans may be developed if we fall behind our target in the attainment of winning bids. Here we often key to the Competitive Advantage Fee ("CAF") dual evaluator.

This generalized approach has proven successful in increasing profitability in a complex business area where sales orders are based on competitive bidding.

3. Guidelines for Evaluating Bidding Procedures at All Levels of Difficutly

The purpose of this section is to suggest models and guidelines within which most bidding procedures must minimally be evaluated.

The underlying uncertainty at the beginning of and during the bidding period involves the market price p_j for part j and estimated annual volume k_j. Normally with bigger k_j's, our unit costs are lower because of relatively less time for setup vis-a-vis running.

In §1, the levels of difficulty (α)—(δ) were introduced and defined by the amount of the complexity in the data available. For each level of difficulty the following basic questions must be resolved in evaluating any bidding procedure.

(a) Which products are "good parts" for the plant, in the sense of physical production?

(b) What is an acceptable mix of products which load up our equipment to an agreed-upon level?

(c) What measures are available for determining the marginal return of a part to be bid upon at a given time with respect to future foregone opportunities associated with its required resource usage?

(d) What measures are there for evaluating our equipment in the market place? Are these measures powerful enough to initiate technological-production change?

(e) For any on-hand product, how can we measure its value as a contributor to maximizing overall C.O.D.?

(f) How can a bidding procedure incorporate differing order sizes and their effect on down time, i.e., setup time?

The purpose of these questions is to illustrate what should be required of a bidding procedure at each level of sophistication. In the next section we restrict attention to the first level of difficulty (α) of the bidding environment. It is proposed here that any ad hoc bidding procedure be evaluated as to how it performs at this level. Historically one might expect alternative bidding procedures to take forms such as the following ones.

3.1 *Ad Hoc Bidding Procedures*

3.1.1 Use mark-up cut-off points to differentiate between good and bad parts.

3.1.2 Compute accounting costs on a per unit basis and include a profit percentage perhaps based on an internal rate of return on investment. Include a penalty for small volume parts.

3.1.3 Use industrial engineering in order to determine easy-to-make parts as well as hard-to-make parts by referring to a list of attributes which inform us on the physical nature of the particular part. Thus, we can tell by our BASIC ATTRIBUTE LIST precisely how to load up our machines allowing adequate time for setup, lunch, coffee-breaks, etc. Thus, we will be assured that when it comes time to produce the parts we have won, we can produce them. Consistent then with the physical approach, we are justified in cutting prices across the board on those parts which fit in nicely according to the BASIC ATTRIBUTE LIST.

3.1.4 A statistician suggests running a least-squares regression of selling price against (1) labor and (2) material and using the computed coefficients a and b in a formula:

$$\text{BID PRICE} = a \times \text{LABOR} + b \times \text{MATERIAL}$$

3.2 *Bidding Environment for First Level of Difficulty* (α)

In order to evaluate any bidding procedure, such as those above, the first step necessarily is to assess its performance at the First Level of Difficulty.

3.2.1 *First Level of Difficulty.* The assumptions are the following. At the beginning of the year (1) all products are known, i.e., the "blue-prints" are on hand, (2) each market price p_j is known for part j, (3) each estimated annual volume k_j is known. Such a situation could be realized in practice if all bids were let initially with suggested prices (as when selling a house), with ties among competitors implying a split-up of the estimated annual volume among them.

More importantly, any bidding procedure that is worthwhile should perform satisfactorily in this environment with regard to questions such as (a) through (f), because of the following reasons.

(i) When market equilibrium or stability occurs, it will not be difficult for a competitor to estimate the market price of any given part as well as its volume. For example, company personnel over the years have developed quite a talent for estimating what a product "will go for".

(ii) A second reason for expecting worthwhile performance of any bidding procedure at the First Level of Difficulty is that it does not make sense mathematically to have a valid procedure in general which is invalid for the mathematically simplest case.

3.3 *A Linear Programming Model for the First Level of Difficulty* (α)

For convenience we shall use only two restricted resources, having c_1 and c_2 hours available for the year. Let x_j denote the amount of contract j desired which is to be either 0 or the customer requested amount k_j. Thus x_j is to be determined at 0 or k_j by the model. Let a_{1j} be the amount of time required on machine 1 to produce a unit of contract j and similarly a_{2j} is defined for contract j on machine 2. For convenience we assume 12 hours of setup are permissible for each contract which uses machine 1 and 6 hours of setup are permissible for each contract that uses machine 2.

The direct cost d_j for contract j consists of labor, material and packaging. The symbol c_i represents total available machine hours. We shall develop a simple linear programming model which incorporates the k_j's and hours spent on setup for machines 1 and 2.

We also assume that a product is produced on one and only one of machines 1 and 2. From the data input viewpoint this simply means not both a_{1j} and a_{2j} are positive, but at least one is.

We now consider the following linear program (I), which seeks to maximize total C.O.D.

Program (I)

Compute

$$\max \sum_{j=1}^{N} (p_j - d_j)x_j \quad \textit{for all} \quad x_j, j = 1, 2, \cdots, N, \quad \textit{and } S_1, S_2$$

subject to

(1) $$\sum_{j=1}^{N} a_{1j}x_j + S_1 \leq c_1 \qquad \text{(Machine \#1)},$$

(2) $$\sum_{j=1}^{N} a_{2j}x_j + S_2 \leqq c_2 \qquad \text{(Machine \#2)},$$

(3) $$\sum_{j=1}^{N} (12/k_j)x_j - S_1 \leqq 0 \qquad \text{(Setup \#1)},$$

(4) $$\sum_{j=1}^{N} (6/k_j)x_j - S_2 \leqq 0 \qquad \text{(Setup \#2)},$$

(5) $$x_j \leqq k_j, \; x_j \geqq 0, \qquad j = 1, 2, \cdots, N.$$

The variables S_1 and S_2 denote setup time on machines 1 and 2 respectively. We assume that there is at least one profitable contract which can be produced on machine #1 and at least one profitable contract which can be produced on machine #2. It then follows that constraints (3) and (4) occur as equalities at an optimal solution, see Appendix 2. Program (I) is an extension of the well-known machine loading model. While it is linear, we show in Appendix 2 that the variable x_j will be either 0 or k_j for at least N–2 contracts in an optimal solution.

3.3.1 *Optimal Bid Price Rules for Level* (α). To each of the constraints in (1)–(5) is associated a dual variable which appears in the dual linear program to (I). Because of their interpretations, we shall call these variables "dual evaluators". Let ω_i be the dual evaluator associated with constraint i, $i = 1, 2$, termed a "machine hour rate". Let m_i be the dual evaluator associated with constraint i, $i = 3, 4$. m_i the "setup constraint" evaluator. Finally, let f_j, $j = 1, 2, \cdots, N$, be the dual evaluator associated with the constraints in (5). It shall be termed the "competitive advantage fee", denoted CAF.

An optimal set of dual evaluators determines an optimal solution to Program (I), and vice versa. Thus, optimal dual evaluators determine which contracts are worthwhile with respect to Program (I). Consequently, we define an *optimal bid price* b_j associated with an optimal set of dual evaluators as follows.

$$(R\alpha) \qquad \begin{aligned} b_j &= d_j + \omega_1 a_{1j} + f_j + \frac{12m_1}{k_j}, \qquad \text{if a machine \#1 contract,} \\ &= d_j + \omega_2 a_{2j} + f_j + \frac{6m_2}{k_j}, \qquad \text{if a machine \#2 contract.} \end{aligned}$$

Then according to linear programming duality theory,

$$b_j \geqq p_j \quad \text{for} \quad j = 1, 2, \cdots, N,$$

and

$$b_j = p_j \quad \text{for contracts } j \text{ to be produced.}$$

3.3.2 *Interpretations and Extensions.* The ω_i evaluators ($i = 1, 2$) are what Seiler in §2 refers to as machine hour rates determined by opportunity cost calculations, while the f_j evaluators are called "CAF's" where the limiting factor here is essentially the market reality rather than production constraints.

When the bid rule ($R\alpha$) is applied to the contracts j, $j = 1, 2, \cdots, N$, under the first level of difficulty (α), then a contract is awarded whenever $b_j = p_j$ and lost whenever $b_j > p_j$. It follows from the duality theory of linear programming that at the conclusion of the bidding process, total C.O.D. of Program (I) is maximized.

Rule ($R\alpha$) has been applied in conditions that only approximate roughly the first level of difficulty assumptions (α). In practice, these models contain more than 1500 contracts which may be produced in alternative ways involving a dozen or more machines. Also, in practice, contingency plans and adjustments were made before submission of a final bid price. These modifications keyed on the CAF dual evaluators and involved the judgment of management. Thus when needed, conditional and adaptive revisions of the formal bid price were made according to new information.

We emphasize that these *large variable* linear programming models are to be interpreted and implemented in an *adaptive* manner. Therefore, beginning with an initial set of evaluators, it is important that these should be periodically revised during the bidding period to take advantage of the conditional status of the resource capacity remaining unallocated and any possible price information. To accomplish this at time t, we partition last year's data into two sets of contracts, J_t and $J - J_t$, where J_t is representative of the contracts seen during the current bidding period up to time t.[1] Letting c_i^t be the total amount of resource i allocated at time t to the contracts won up to that time, we can now solve

$$\max_{x_j} z = \sum_{j \in J - J_t} (p_j - d_j) x_j$$

subject to

$$\sum_{j \in J - J_t} a_{ij} x_j \leqq (c_i - c_i^t), \qquad i \in I,$$

$$0 \leqq x_j \leqq k_j, \qquad j \in J - J_t,$$

where I is the index set for the resources and where, for simplicity, we have omitted the setup coefficients from the constraints.

Analogous to the definitions above, ω_i^t is the optimal dual evaluator associated with the resource constraint i. The ω_i^t are then the machine hour rates to be applied in time period t.

When interpreting and implementing these models in the bidding environment, additional constraints were developed and used as ways for handling certain risks. For example, analogous to notions of "payback" and "horizon posture" (see [2]), we developed market penetration and C.O.D. coverage constraints to be applied at each bidding stage. These measures are used as filters for risks which probably would not be acceptable to the firm in its bidding policies. Perhaps few firms would be willing to lose essentially all bids for many time periods with the hope that the jackpot will come near termination of the entire bidding season and such that the target C.O.D. will be attained. Analogous remarks apply to the interpretation of "interim" posture constraints for the firm. Associated with both of these constructs, since they are used here in constraint form, economic dual evaluators provide useful measures for the costs and implications of these devices, which are designed in part to assist in implementing the linear programming (large scale) models as adaptive conditional variants of an a priori LP model. This methodology appears to yield insight into the dynamic nature of relative competitive values of restricted resources over time thereby providing systems measures for production and capacity alterations.

4. Responses to Bidding Environments of Higher Difficulty (δ)

We now discuss responses to problems encountered at the more realistic bidding environment level (δ) described in §1. In doing so at this time we have skipped over intermediate levels (β) and (γ). Their development, however, is given in Appendix 1, and the optimal rules derived for these intermediate levels suggest that the general bid rule form (R) given below is sensible for implementation at level (δ).

(R) OPTIMAL BID PRICE = DIRECT COSTS + OPPORTUNITY COSTS + COMPETITIVE ADVANTAGE FEE

[1] We may of course incorporate whatever scarce market price data that are available to the firm for the current year into the LP model in an analogous fashion.

where the opportunity costs are the restricted resource charges, and the competitive advantage fee is the nonnegative additional charge if the firm is especially competitive for a certain contract.

Because of the large number of contracts involved (over 1500) and their production interdependencies, we use the linear programming model developed at the very first level of difficulty as an approximation of the complex environment (δ). Depending on the amount of overall stability and validity of characteristics (α), the first level rule becomes dominant over others according to §3. Hence the associated bid price rules of §3.3.1 are used with management judgment and market intelligence in an adaptive and conditional manner as described in §2.

Thus, for level (δ), we are suggesting bid pricing rules for situations where managements know much less about which contracts are to appear and when, but where the current market is expected to react similarly to that of previous years. Although in many cases one would want to maximize the expected marginal return from each contract, we have sought a method here which determines or approximates the marginal return of a contract with respect to future foregone opportunities for restricted resource usage.

In a restricted resource environment, the question of the capacity of resources over which to allocate requires definition. By *capacity* we mean a balance or mix between the physical output of the equipment and the manpower required to support that output. This balance normally will be mutually agreed upon both by management and labor (e.g., unions). In this and the following Appendices we shall assume that all our bidding procedures and methodology are based on capacity operations. This implies that certain physical operations or their combinations are performed in a manner that may not necessarily be done for a less than capacity operation. We bid at capacity because that is where the firm's competitive advantage is most likely to be. For example, at a capacity operation, the firm may be willing to double labor costs in order to fully utilize restricted physical resources, such as machines, thereby obtaining a high rate of competitive production.

We do not explicitly include in our models an evaluation of such alternatives intrinsic to the notion of capacity defined as a mix. Our thrust here is upon using well-known computational methods and their attendant economic evaluators to explore alternate bidding strategies. It is possible, however, to use these methods for a more detailed study of the nature of "capacity" *per se* and for evaluation of various combinations of labor and physical productive equipment.

The firm does not know which contracts are to appear during the bidding period, but feels that the market in total will be similar to that of the preceding year. Several authors, including Mobley [16], have suggested that resources should *a priori* be evaluated in proportion to their past contribution in a similar market. There appear to be two objections to this. First, the method for determining the individual contribution of a resource in an environment characterized by multiple and/or alternative resource usages per contract remains unanswered. Second, the firm's performance in the past year's market may not have committed resources efficiently; therefore, pricing resources on this basis would be equivalent to requesting a similar policy for the upcoming period.

A proposal of S. A. Tucker [25], [26] for evaluating resources might be to use the relative costs of acquiring the resource capacities as proportional parts of the budgeted contribution over direct cost (C.O.D.) to be allocated. For example, one might calculate a weighted average:

$$k_i = \left(\frac{w_i}{\sum_{i \in I} w_i}\right) \frac{\text{C.O.D.}}{c_i}, \qquad \text{for each resource } i,$$

where w_i is the cost of having obtained c_i units of restricted resource i. However, it is difficult to incorporate such factors as technological obsolescence and inefficient usage of resources in this method. In addition, it may bear no relationship to the value that the market may be placing on resources. Littlechild [14], however, introduces methods which explicitly contrast market evaluations of restricted resources with their acquisition costs. He shows not only that prices should be set to fully use capacity, but that capacity should be increased until the opportunity costs it earns covers its purchase cost (w_i above) and production cost. Quoting [14], "Rather, *price should be set to just fully utilize capacity* (subject to price not being below marginal production cost), *and capacity should be chosen so that, over the life of the asset, capacity costs are just recouped.*" We point out that with respect to capacity expansion, our comparisons of alternatives of extra equipment (in practice) were made according to Littlechild's results. The terminology "resource usage charge" which we have employed throughout this paper is analogous to his notion of "opportunity costs that capacity earns." See also Kaplan-Thompson [9] for various rules for overhead allocation stemming from mathematical programming models.

Our emphasis, however, is on models which yield market evaluations and alternative future evaluations while maintaining capacities initially as fixed and regarding their associated costs as sunk. These models must and do possess parametric capabilities available for decisions of additional capacity commensurate with results of Littlechild.

Another unexplored facet of this problem is how to determine the budgeted C.O.D. within the time period of utilization. We shall not discuss this here except to state that the relative resource pricing schemes we develop also provide a framework within which parametric studies of the total C.O.D. can be made. In particular, the bid pricing mechanism is designed to modulate resource evaluations in order to force contact bid winnings to occur, enabling the resulting job mix to best utilize the total capacity of the restricted resources.

One is faced with deciding how to measure the opportunity costs of resources. Contracts must first bid against future contracts the firm may face for the use of resources while simultaneously competing in the outside market. Our models (including the dynamic extensions as given in Appendix 1) developed in the previous sections have led to a bid price rule of the general form (R) above.

In the following appendix we derive various forms of the rule (R) determined mathematically for the levels of difficulty (β) and (γ).

Appendix 1

On Sequential Probabilistic Multi-Contract Bid Pricing Models

The purpose of this appendix and some of the developments of our earlier manuscript [11] is to develop various bidding rules under various mathematical assumptions on the complexity of the bidding process. We first introduce a probability distribution on the minimum winning bid price and seek to (1) maximize expected markup, and later (2) to maximize the probability of attaining a given aspiration level. We then introduce a probability distribution on the restricted resource itself, thereby making it a random variable.

We do not mean to imply that these extensions have been implemented by verifying the assumptions of any one of them in practice. Rather, the mathematical form of the optimal bid price rules of each has reinforced the use of the form derived at the "first level of difficulty", namely the general bid rule (R) given in the text:

OPTIMAL BID PRICE = DIRECT COSTS + OPPORTUNITY COSTS + COMPETITIVE ADVANTAGE FEE.

This appendix details the mathematical form of this rule for the levels of difficulty (β) and (γ). The levels (α) and (δ) have been discussed in the text.

1. The Market Price as a Random Variable: Level of Difficulty (β)

We shall now assume that we know a_{ij}, d_j for all $j \in J$ as well as c_i for all $i \in I$. However we do not know the price, p_j, at which the contract j can be won in the market place. Instead we assume the existence of a known function, $g_j(b_j)$, for our bid price, b_j, such that

$$g_j(b_j) = \text{probability of winning contract } j \text{ with bid price } b_j$$
$$= \int_{p_j=b_j}^{\infty} dF_j(p_j) = 1 - F_j(b_j)$$

where $F_j(p_j)$ is the cumulative distribution function of the minimum bid price, p_j, of all other competitors. The assumption of such a function is common in the bidding literature. Several such random variables and the associated distributions have been used to introduce market uncertainty into bidding models. See Stark and Mayer [24] and Stark [23], and also the work of Christenson, Edelman, Friedman, Dean, Hanssman and Rivett, Howard, Ortega-Reichert, Robinson, Rothkopf and Simmonds all referenced in Stark [23].

Our purpose is not to propose a specific distribution for $F_j(p_j)$ but to suggest that the assumption of the existence of such does embed some measure of the uncertainty of the market into the model. Actually there is still active discussion of the distributions $F_j(p_j)$; see, for example, Stark [22]. Obviously, the competitors do not act according to $F_j(p_j)$; however, this function embodies what we observe the competitive behavior to be as a function of our submitted bid prices.

The use of such a function has been criticized because it does not account for the reaction that a competitor may have to our bid price strategy during the bidding period. Yet, as stated previously, the competitors in our sequential problem know only that they will win or lose a contract—no more. Thus, although it may be theoretically appealing to speak of a competitor reaction, if one cannot observe or intuit a specific competitor's state of nature, one can hardly consider his reaction strategies that are dependent upon such a state. For both this reason and the indications that tacit agreements concerning markup percentages do exist within many market places, we maintain that $F_j(p_j)$ is a reasonable characterization of the behavior of the competition in an imperfect, but essentially stable market. It is possible to develop optimal bid price rules when an adaptive version for the $F_j(p_j)$ is used. Some of these extensions were set forth in [11].

2. The Bidding Process Under Price Uncertainty—Maximizing Expected Markup (β)

Given $g_j(b_j)$, which may reflect uncertainty concerning both the cost of fulfilling the contract and level of market prices, our bidding process is a series of Bernoulli trials where

$$\Pr(\text{winning contract } j \mid b_j) = g_j(b_j), \qquad \Pr(\text{losing contract } j \mid b_j) = 1 - g_j(b_j).$$

However, when a restricted resource has been allocated to contracts won to the extent that insufficient capacity remains for a future contract, then that contract cannot be bid upon (ignoring outside contracting, overtime, etc.).

Let us define the following:

$$\begin{aligned}
J_k &= \{j \mid \text{contract } j \text{ has been won and contract } k \text{ is to be bid upon next}\},\\
c_{ik} &= \text{quantity of resource } i \text{ unallocated after the first } (k-1) \text{ bids}\\
&= c_i - \textstyle\sum_{j \in J_k} a_{ij}, \; i \in I, \text{ where } I \text{ indexes available resources},\\
C_k &= \{c_{ik}, i \in I\}, \text{ a vector of unallocated resources after } (k-1) \text{ bids},\\
a_k &= \{a_{ik}, i \in I\}, \text{ a vector of resource requirements for the } k\text{th bid},\\
C_k - a_k &= \{(c_{ik} - a_{ik}), i \in I\}.
\end{aligned}$$

Now assuming that the objective for the firm is to maximize the total contribution over cost (C.O.D.) during the bidding period, we can describe the bidding process by the following functional equations:

$$\begin{aligned}
\Psi_k(C_k) &= \text{maximum expected C.O.D. for the contracts } k, \cdots, n \text{ given the unallocated resource capacity } C_k\\
&= \max_{b_k} \{g_k(b_k)[b_k - d_k + \Psi_{k+1}(C_k - a_k)] + [1 - g_k(b_k)]\Psi_{k+1}(C_k)\}\\
&\quad \text{if } \; C_k - a_k \geqq 0\\
&= 0 + \psi_{k+1}(C_k) \text{ otherwise}, \quad \text{for } k = 1, \cdots, n-1.
\end{aligned}$$

Now for $k = n$ we define

$$\begin{aligned}
\Psi_n(C_n) &= \operatorname{Max}_{b_n} \{g_n(b_n)[b_n - d_n]\}, \quad \text{if } \; C_n - a_n \geqq 0,\\
&= 0. \qquad\qquad \text{otherwise.}
\end{aligned}$$

Following the definitions above, observe that $\Psi_1(C_1)$ is the maximum expected markup from the bidding process, given the availability of c_i units of the restricted resource i. Thus if "OH" is the total indirect overhead associated with the time period of production of the awarded contracts, then the maximum expected profit is $[\Psi_1(C_1) - \text{OH}]$.

We need the following definitions:

(A) $\beta_k = \inf \{b_k \mid g_k(b_k) = 0\}$,

(B) $\lambda_k(C_k) = \Psi_{k+1}(C_k) - \Psi_{k+1}(C_k - a_k)$.

The price β_k is the bid price for contract k above which it cannot be won. In (B), $\lambda_k(C_k) \geqq 0$, and is a measure of the value of the resources a_k if contract k is lost. We shall show that $\lambda_k(C_k)$ may be viewed as a resource usage charge to contract k in forming its bid price. We need, however, some regularity conditions on the functions $g_k(b_k)$ and so we review one of these called *unimodality*.

A differentiable function $h(t)$ for $a \leqq t \leqq b$ will be termed *unimodal* whenever $h'(t^*) = 0$ for some t^*, $a \leqq t^* \leqq b$, is equivalent to $h(t^*) = \max \{h(t) \mid a \leqq t \leqq b\}$.

We now derive Theorem I which gives optimal bid rules under the "level of difficulty" defined in this section, which corresponds to (β) described in the text.

THEOREM I. *Assume the following four properties hold.*

1. $g_k(b_k)$ *is a continuously differentiable function*, $0 \leqq b_k \leqq \beta_k$, $g_k(b_k) = 0$ *for* $b_k \geqq \beta_k$ *where* β_k *is defined in* (A) *above, and* $g_k(b_k) > 0$ *for* $0 \leqq b_k < \beta_k$.

2. *For any fixed* x, $0 \leqq x < \beta_k$, *the function* $g_k(b_k)(b_k - x)$ *is unimodal for* $x \leqq b_k \leqq \beta_k$.

3. $d_k + \lambda_k(C_k) < \beta_k$, *where* $\lambda_k(C_k)$ *is defined in* (B) *above, and* d_k (*direct cost*) > 0.
4. $C_k - a_k \geqq 0$.
Then an optimal bid price b_k^* *exists and satisfies*

$$b_k^* = d_k + \lambda_k(C_k) - g_k(b_k^*)/g'_k(b_k^*)$$

where $g'_k(b_k^*)$ *denotes the derivative of* g_k *at* b_k^*.

PROOF. By definition we have that

$$\Psi_k(C_k) = \Psi_{k+1}(C_k) + \max_{b_k} \{g_k(b_k)[b_k - d_k - \lambda_k(C_k)]\}.$$

Since $\Psi_{k+1}(C_k)$ is independent of b_k it suffices to investigate

$$h_k(b_k) = g_k(b_k)[b_k - d_k - \lambda_k(C_k)],$$

which is continuously differentiable for $0 \leqq b_k \leqq \beta_k$. By inspection we see:

(a) $h_k(b_k) = 0$ for $b_k \geqq \beta_k$,
(b) $h_k(b_k) \leqq 0$ for $0 \leqq b_k \leqq d_k + \lambda_k(C_k)$, and
(c) $h_k(b_k) > 0$ for $d_k + \lambda_k(C_k) < b_k < \beta_k$,

where (c) follows from assumption 1 and the form of $h_k(b_k)$. Therefore h_k assumes a maximum at b_k^*, $d_k + \lambda_k(C_k) < b_k^* < \beta_k$. But $h_k(b_k)$ is unimodal for $d_k + \lambda_k(C_k) \leqq b_k \leqq \beta_k$ by assumption 2, and therefore its derivative at b_k^* is zero, i.e.,

$$0 = h'_k(b_k^*) = g_k(b_k^*) + g'_k(b_k^*)(b_k^* - d_k - \lambda_k(C_k)).$$

However since $g_k(b_k^*) > 0$, it follows that $g'_k(b_k^*) \neq 0$ and therefore

$$b_k^* = d_k + \lambda_k(C_k) - g_k(b_k^*)/g'_k(b_k^*)$$

completing the proof of Theorem I.

Remark. The unimodality assumption is weaker than earlier concavity and monotonicity assumptions we used for Theorem I, as pointed out to us by a referee. We use the previous assumptions in Theorem II.

The following Corollary gives the optimal bid price when $d_k + \lambda_k(C_k) \geqq \beta_k$.

COROLLARY. *Given assumptions* 1, 2, *and* 4 *of Theorem* I *and* 3′: $d_k + \lambda_k(C_k) \geqq \beta_k$. *Then an optimal bid price* b_k^* *is given by* $b_k^* \geqq \beta_k$, *in particular,* $b_k^* = d_k + \lambda_k(C_k)$.

PROOF. Properties (a) and (b) in the proof of Theorem I are still valid in this case. Hence $h_k(b_k) < 0$ for $0 \leqq b_k < d_k + \lambda_k(C_k)$, and $b_k < \beta_k$. Since

$$\beta_k \leqq d_k + \lambda_k(C_k),$$

it follows that any $b_k^* \geqq \beta_k$ is optimal. Thus, a negative markup is avoided. Q.E.D.

Thus we see that our optimal bidding strategy is essentially to charge contract j with resource usage charge $\lambda_j(C_j)$, the opportunity cost of the a_j resource units given current capacity of C_j. Then, if there is a positive probability still associated with winning the contract, one calculates b_j^* according to Theorem I. Letting

$$\gamma_j^* = -(g_j(b_j^*)/g'_j(b_j^*)),$$

the conclusion of Theorem I becomes:

$$(R\beta) \qquad b_j^* = d_j + \lambda_j(C_j) + \gamma_j^*.$$

This rule is of the same form developed for the first level of difficulty (α) in the text and may be compared with rule (R) given in §4 in verbal form.

Here however, the resource usage charge $\lambda_k(C_k)$ is not in general a linear function of a_k. Moreover, these charges are conditional upon the resource capacities C_k remaining unallocated after the first $k - 1$ bids.

We discuss now the situation when the $(k - 1)$th optimal bid price is higher than the going market price. In practice it is sometimes feasible to bid at a price lower than the optimal bid price but above the direct cost of the contract. At least a contribution to overhead is obtained in this way, but at the expense of using up resources which could have been better employed. Acceptance of such a contract would be especially feasible if there were possibilities of recovering C.O.D. in the future.

When the $(k - 1)$th contract is bid at its optimal bid price and loses, then $C_k = C_{k-1}$. Attention now is centered on how to bid for contract k which uses resources a_k. We assume contract k is identical to $k - 1$ and that the probability of winning is the same for each.

Thus,

(C) $a_{k-1} = a_k$, $d_{k-1} = d_k$, $g_{k-1}(b) = g_k(b)$ for all b.

We assume also that the resource usage charge has not increased because of the loss of contract $k - 1$, i.e.,

(D) $\lambda_k(C_{k-1}) \leqq \lambda_{k-1}(C_{k-1})$.

Under the above conditions we now relate the kth bid price to the $(k - 1)$th bid price as follows.

THEOREM II. *Assume that the following properties hold.*

1. $g_k(b_k)$ *is a continuously differentiable concave strictly decreasing function for* $0 \leqq b_k \leqq \beta_k$, *and* $g_k(b_k) = 0$ *for* $b_k \geqq \beta_k$, *while* $g_k(b_k) \geqq 0$ *for* $0 \leqq b_k \leqq \beta_k$.
2. $d_j + \lambda_j(C_j) < \beta_j$, $j = k - 1, k$.
3. $C_j - a_j \geqq 0$, $j = k - 1, k$.
4. (C) *and* (D) *above.*

Then if the optimal bid b_{k-1}^* *for contract* $k - 1$ *loses, the optimal bid* b_k^* *for contract* k *satisfies* $b_k^* \leqq b_{k-1}^*$.

PROOF. Defining $h_k(b_k)$ as in the proof of Theorem I, it follows from 1 above that h_k is a concave function for $d_k + \lambda_k(C_k) \leqq b_k \leqq \beta_k$ since

$$h''_k(b_k) = 2g'_k(b_k) + g''_k(b_k)(b_k - d_k - \lambda_k(C_k)) \leqq 0$$

due to the fact that $g_k(b_k)$ is concave decreasing. Thus, $h_k(b_k)$ is unimodal in this situation.

Therefore, Theorem I applies to yield optimal bid price rules:

(II.1) $b_{k-1}^* = d_{k-1} + \lambda_{k-1}(C_{k-1}) - g_{k-1}(b_{k-1}^*)/g'_{k-1}(b_{k-1}^*)$,

(II.2) $b_k^* = d_{k-1} + \lambda_k(C_{k-1}) - g_{k-1}(b_k^*)/g'_{k-1}(b_k^*)$,

using (D) and $C_{k-1} = C_k$ stemming from the loss of bid $k - 1$.

Hence using assumption (D),

(II.3) $b_k^* - b_{k-1}^* = \lambda_k(C_k) - \lambda_{k-1}(C_{k-1}) + (\Delta_k - \Delta_{k-1}) \leqq \Delta_k - \Delta_{k-1}$, where $\Delta_k = -g_k(b_k^*)/g'_k(b_k^*)$. However, upon checking derivatives it follows that Δ_k is a decreasing function of b_k. Therefore, if $b_k^* > b_{k-1}^*$ then $\Delta_k - \Delta_{k-1} < 0$ and a contradiction of (II.3) is obtained. Hence $b_k^* \leqq b_{k-1}^*$, Q.E.D.

3. The Bidding Process Under Price Uncertainty—Maximizing the Probability of Attaining a Given Aspiration Level

We continue to investigate level of difficulty (β) as in the previous section but now rather than maximizing the total expected markup, the goal is to maximize the probability of gaining a total markup of, say z. Thus we are adopting a satisficing approach where z is a target value, e.g., budgeted contribution over direct (C.O.D.).[2]

Letting

$$z_k = z - \sum_{j \in J_k} (b_j - d_j)$$
$$= \text{total contribution still to be attained after the first } (k-1) \text{ bids,}$$

we can again describe our bidding process by a set of functional equations:

$\Psi_k(C_k, z_k)$ = maximum probability that z will be attained by the end of the bidding period, given the remaining resource capacity of C_k and target of z_k remaining after the first $(k-1)$ bids,

$$\begin{aligned} &= \text{Max}_{b_k}\, g_k(b_k)[\Psi_{k+1}(C_k - a_k, z_k - b_k + d_k)] \\ &\quad + [1 - g_k(b_k)]\Psi_{k+1}(C_k, z_k), && \text{if } C_k - a_k \geqq 0, \quad z_k > 0, \\ &= 0 + \Psi_{k+1}(C_k, z_k), && \text{if } C_k - a_k \ngeqq 0, \quad z_k > 0, \\ &= 1, && \text{if } z_k \leqq 0, \text{ for } k = 1, \cdots, n-1. \end{aligned}$$

For $k = n$, we define

$$\begin{aligned} \Psi_n(C_n, z_n) &= \max_{b_n} g_n(b_n) && \text{if } C_n - a_n \geqq 0, \quad z_n > 0, \quad z_n - b_n + d_n \leqq 0, \\ &= 0 && \text{if } C_n - a_n \ngeqq 0, \quad z_n > 0, \\ & && \text{or if } C_n - a_n \geqq 0, \quad z_n - b_n + d_n > 0, \\ &= 1 && \text{if } C_n \geqq 0, \quad z_n \leqq 0. \end{aligned}$$

Thus $\Psi_1(C_1, z)$ is the maximum Pr{total markup $\geqq z$} given the availability of c_i units of resource i. Also letting

$$\lambda_j(C_j, z_j, b_j) = \Psi_{j+1}(C_j, z_j) - \Psi_{j+1}(C_j - a_j, z_j - b_j + d_j),$$

we have the following theorem.

THEOREM III. *Assume the following properties hold.*

1. *Assumption* 1 *of Theorem* II.
2. $\lambda_k(C_k, z_k, \beta_k) < 0$.
3. $C_k - a_k \geqq 0$.

Then an optimal bid price b_k^* *exists,* $d_k < b_k^* < \beta_k$. *When the*

$$\Psi_{k+1}(C_k - a_k, z_k - b_k, d_k)$$

is differentiable at b_k^*, *then*

$$\lambda_k(C_k, z_k, b_k^*) = \left.\frac{\partial \lambda_k(C_k, z_k, b_k)}{\partial b_k}\right|_{b_k^*} \left[\frac{-g_k(b_k^*)}{g'_k(b_k^*)}\right].$$

PROOF. $\Psi_k(C_k, z_k) = \Psi_{k+1}(C_k, z_k) + H(b_k)$, where

$$H(b_k) = \max_{b_k} \{-g_k(b_k)\lambda_k(C_k, z_k, b_k)\}.$$

$\Psi_k(C_k, z_k)$ can be shown to be a concave, nonincreasing function of z_k by induction, and hence is a continuous function of z_k. Thus, $\Psi_{k+1}(C_k - a_k, z_k - b_k + d_k)$ is continuous in b_k, and hence so is $\lambda_k(C_k, z_k, b_k)$. Therefore $H(b_k)$ assumes a maximum at b_k^*, $d_k \leqq b_k^* \leqq \beta_k$. However since $\lambda_k(C_k, z_k, \beta_k) < 0$, it follows that $H(b_k) > 0$

[2] See Charnes-Stredry [6] for a further discussion of aspiration criteria.

in a neighborhood of β_k. Furthermore since

$$\lambda_k(C_k, z_k, d_k) = \Psi_{k+1}(C_k, z_k) - \Psi_{k+1}(C_k - a_k, z_k) \leqq 0$$

it follows that $H(d_k) \leqq 0$. Hence $d_k < b_k^* < \beta_k$. When $\Psi_k(C_k - a_k, z_k - b_k, d_k)$ is differentiable at b_k^*, it follows that

$$\frac{\partial}{\partial b_k}[-g_k(b_k)\lambda_k(C_k, z_k, b_k)]|_{b_k^*} = 0$$

that is,

$$\lambda_k(C_k, z_k, b_k^*) = \frac{\partial \lambda_k(C_k, z_k, b_k)}{\partial b_k}\bigg|_{b_k^*}\left[\frac{-g_k(b_k^*)}{g_k'(b_k^*)}\right].$$

4. Price Uncertainty—Market Reaction Level of Difficulty (γ)

It is intuitively appealing to be able to analytically specify through $g_j(b_j)$ the market reactions that could be expected as a result of various bidding strategies. Since we are unaware of any specific competitor's state of nature, we must assume a market reaction dependent upon the only information that we possess—our own relative standing vis-a-vis (a) bids won, (b) total dollars won, (c) market penetration and other attributes.

Thus we can assume the existence of:

$g_j(b_j \mid \varnothing_j) = \Pr\{$winning contract $j \mid$ bid price b_j, market price level $\varnothing_j\}$,

where $\varnothing_j$ is a function of our policies and/or performance for the first $j - 1$ contracts. Possible measures of such a function may include:

(a) the fraction of the total contract value of the $j - 1$ contracts won,

(b) the fraction of the number of contracts won,

(c) the average bid markup,

(d) the deviation from the market penetration achieved in past bidding periods.

The difference between $g_j(b \mid \phi_j)$ and an adaptive estimation process of $g_j(b_j)$ of §2 (as, for example, in [11], §8) may be surmised if we assume that we win the first, say, l bids. As a reasonable market reaction, we would expect $g_{l+1}(b_{l+1} \mid \phi_{l+1})$ to exhibit the property that the market bid prices are lower, forcing a lower b_{l+1} to attain a probability p of winning contract $l + 1$ than if we had lost the first l bids. Now an adaptive estimation process for the $g_j(b_j)$ (see [11]) might indicate that the market is bidding high and thus permit us to raise b_{l+1} to attain probability p of winning relative to such a b_{l+1} had we lost the first l bids. Thus although the analysis of the market price level relative to our bid prices is identical in both cases, the adaptive model assumes it will remain as is and moves to take advantage of this while the reaction model assumes it cannot remain so and moves in anticipation of its expected correction.

We note that a strategy of placing otherwise noncompetitive bids for certain contracts can encourage higher market prices that may lead to a greater overall expected return for the entire bidding period. We believe that under the conditions of market reactions it should be possible to describe the bidding process by a set of functional equations where ϕ_j becomes an additional state variable, analogous to the earlier dynamic programming models of §2.

5. Resource Allocation Uncertainty: Level of Difficulty (γ)

In the previous sections we have assumed that the amount of resource i required to fulfill contract j at a later time period was a known constant, a_{ij}. If instead we assume that a_{ij} is a random variable, then, in general, we would expect the direct costs, d_j, to be probabilistic. Also, since the estimate of the probability of winning contract j at bid price b_j is usually based upon d_j, it is apparent that a characterization through b_j alone is insufficient. However, the more important question is how to interpret the restricted resource capacities, given random future allocations.

For the purpose of disposing of the first two complications, we shall assume that d_j is a random variable with c.d.f. $F_j(d_j)$ where our $g_j(\cdot)$ function now appears as

$$g_j(b_j \mid d_j) = \Pr\{\text{winning contract } j \mid \text{bid price } b_j, \text{ direct costs } d_j\}.$$

Our sequential process does not reveal any information about the distribution of the a_{ij}'s since the resource allocation will be made in the future. Of course we do learn which contracts are won, thereby defining J_k, which updates our knowledge of the distribution of $\sum_{j\in J} a_{ij}$.

In order to express our constraints on the restricted resources we must use the concepts of probabilistic constraints. One technique for handling constraints involving random variables is chance-constrained programming[3] in which one replaces

$$\sum_{j\in J_n} a_{ij} \leqq C_i, \qquad i \in I,$$

with

$$\Pr\{\sum_{j\in J_n} a_{ij} \leqq C_i \mid J_n\} \geqq \alpha_i, \qquad \forall i \in I,$$

where α_i is a given constant, $0 \leqq \alpha_i \leqq 1$, $\forall i \in I$.

Theorem IV and Lemma 2 which follow are used to compute the form of an optimal bid price rule, which is related to those developed earlier at the first level of difficulty, and in the more complicated situation of §2 of this appendix.

THEOREM IV. *Given* 1. $\bar{j}_j = \min d_j$ *such that* $F(d_j) = 1$

2. a_{ij}*'s independent random variables* $\sim N(\mu_{ij}, \sigma_{ij}^2)$, $\mu_{ij} \geqq 0$
3. $\alpha_i \geqq 1/2$
4. $Z^{-1}(y)$ *defined as min* u *such that* $Pr\{t \leqq u\} \geqq y$ *where* $t \sim N(0, 1)$ *then*

$$\Pr\{\sum_{j\in J_n} a_{ij} \leqq c_i \mid J_n\} \geqq \alpha_i$$

if and only if

$$c_i - \sum_{j\in J_k} \mu_{ij} - \mu_{ik} \geqq Z^{-1}(\alpha_i)\sqrt{}(\sum_{j\in J_k} \sigma_{ij}^2 + \sigma_{ik}^2)^{1/2}, \qquad g_k(b_k \mid \bar{j}_k) = 0 \qquad k = 1, \cdots, n.$$

PROOF. $\Pr\{\sum_{j\in J_n} a_{ij} \leqq c_i) \geqq \alpha_i \Leftrightarrow$

$$\frac{c_i - \sum_{j\in J_n} \mu_{ij}}{(\sum_{j\in J_n} \sigma_{ij}^2)^{1/2}} \geq Z^{-1}(\alpha_i) \Leftrightarrow$$

$$C_i \geq Z^{-1}(\alpha_i)\,(\sum_{j\in J_n} \sigma_{ij}^2)^{1/2} + \sum_{j\in J_n} \mu_{ij}.$$

But since σ_{ij}^2 and μ_{ij} are both nonnegative terms for all j, then any partial sum of these terms must satisfy the inequality, unless $g_j(b_j \mid \bar{j}_j) = 0$, in which case the contract j will not be won and thus, not enter into the sum. Q.E.D.

LEMMA 1. *Given*

1. *assumption* 1, *Theorem* IV
2. *assumption* 3, *Theorem* IV

[3] Introduced by Charnes-Cooper-Symonds [5].

3. a_{ij}'s $\sim N(\mu_{ij}, \sigma_{jj}^{2})$ *with covariance matrix* (σ_{jk}^{2i}), $\mu_{ij} \geqq 0$ *then*

$$\Pr(\textstyle\sum_{j \in J_n} \alpha_{ij} \leqq c_i \mid J_n\} \geqq \alpha_i \quad \text{if}$$

$$c_i - \textstyle\sum_{j \in J_k} \mu_{ij} - \mu_{ik} \geqq Z^{-1}(\sigma_i) \sum_{j \in J_k} \sigma_{ij}^{2} + \sigma_{ik}^{2} + \sum_{j,l \in J_k, j \neq l} \sigma_{jl}^{2i} + \sum_{i \in J_k} \sigma_{jk}^{2i},$$

or

$$g_k(b_k \mid \bar{j}_k) = 0, \qquad k = 1, \cdots, n.$$

Proof. This follows directly from both Theorem V and the definition of the variance of $\sum_{j \in J_n} a_{ij}$.
Thus, for the case where the a_{ij}'s are independent random variables $\sim N(\mu_{ij}, \sigma_{ij}^{2})$ we can amend our notational definition in order that:

$$C_k = \{c_i - \textstyle\sum_{j \in J_k} \mu_{ij}, \forall i \in I\}, \qquad a_k = \{\mu_{ik}, \forall i \in I\},$$
$$\phi_k = \{\textstyle\sum_{j \in J_k} \sigma_{ij}^{2}, \forall i \in I\}, \quad \text{and } \sigma_k^{2} = \{\sigma_{ik}^{2}, \forall i \in I\}.$$

Now due to the deterministic equivalents shown in Theorem V, we can again describe our bidding process by a set of functional equations:

$$\psi_k(C_k.\phi_k) = Max_{b_k}\left[\left[\int_{-\infty}^{\infty} \{g_k(b_k|d_k)[b_k - d_k + \psi_{k+1}(C_k - a_k.\phi_k + \sigma_k^{2})\right] + \right.$$
$$[1 - g_k(b_k|d_k)\,\psi_{k+1}(C_k.\phi_k)\}dF(d_j)] \quad \textit{if} \quad C_k - a_k \geq Z^{-1}(\alpha_i)\sqrt{(\phi_k + \sigma_k^{2})}$$
$$= 0 + \psi_{k+1}(C_k.\phi_k) \quad \text{otherwise}$$

for $k = 1, \cdots, n-1$.
Defining $\Psi_{n+1}(\cdot, \cdot) \equiv 0$ allows us to include the case $k = n$ above.

Theorem V. *Assume that the following properties hold.*
1. *Assumption* 1 of Theorem 1.
2. $\lambda_j(C_j, \phi_j) < \beta_j$,
3. $\alpha_i \geqq \frac{1}{2}, \quad i \in I$,
4. $C_j - a_j \geqq Z^{-1}(\alpha)(\phi_j + \sigma_j^{2})^{1/2}$, read component-wise, $\alpha = (\alpha_1)$

then the optimal bid price, b_j^*, is such that

$$-(\bar{g}_j(b_j^*)/\bar{g}_j'(b_j^*)) = b_j^* - \lambda_j(C_j.\phi_j) - \frac{\int_{-\infty}^{\infty} d_j g'_j(_j^* \mid d_j)dF(d_j b)}{\bar{g}_j'(b_j^*)}$$

where

$$\bar{g}_j(b_j) = \int g_j(b_j \mid d_j)dF(d_j),$$

$$\bar{g}_j'(b_j) = \int g_j'(b_j \mid d_j)dF(d_j),$$

$$g_j'(b_j \mid d_j) = \partial g_j(b_j \mid d_j)/\partial b_j, \quad \text{and}$$
$$\lambda_j(C_j, \phi_j) = \psi_{j+1}(C_j, \phi_j) - \psi_{j+1}(C_j - a_j, \phi_j + \sigma_j^{2}).$$

Proof. From Theorem I we know that $h_k(y \mid d_k)$ is unimodal in y where

$$h_k(y \mid d_k) = g_k(y \mid d_k)[y - d_k - \lambda_k(C_k, \phi_k)].$$

Hence $E_{d_k}\{h_k(y \mid d_k)\} = \int_{-\infty}^{\infty} h_k(y \mid d_k)\, dF(d_k)$ is also unimodal in y. Therefore b_k^*, the optimal bid price, is obtained by setting its derivative equal to zero.

$$\frac{\partial}{\partial b_k}[E_{d_k}\{h_k(b_k \mid d_k\}] = 0 .$$

$$\frac{\partial}{\partial b_k}\int_{-\infty}^{\infty} g_k(b_k \mid d_k)[b_k - d_k - \lambda_k(C_k, \phi_k)]dF(d_k)$$

$$= \int_{-\infty}^{\infty} \{g_k'(b_k \mid d_k)[b_k - d_k - \lambda_k(C, \phi_k)] + g_k(b_k \mid d_k)\}\, dF(d_k)$$

$$= g_k'(b_k)[b_k - \lambda_k(C_k, \phi_k)] - \int_{-\infty}^{\infty} g_k'(b_k \mid d_k)d_k\, dF(d_k) + \bar{g}_k(b_k)$$

Setting this equal to 0 yields the desired result. Q.E.D.

Again our optimal bid price is of the form of that of §2. Another technique for handling constraints involving random variables is linear programming under uncertainty[4] in which one defines a penalty function for violations of the resource constraint and then minimizes the expected value of same. For example, we would define $t_i = c_i - \sum_{j \in J_n} a_{ij}$, a random variable with c.d.f. $G(t_i)$. Let $\theta_i(t_i)$ = cost of procuring additional quantities of i, such that $\theta_i(t_i) \geqq 0$ if $t_i \leqq 0$. Upon defining

$$\psi_{n+1}(C_{n+1}, \phi_{n+1}) = -(\{\textstyle\sum_{i \epsilon I} \int \theta_i(t_i)dG(t_i)\})$$

we can describe the bidding process by the set of functional equations:

$$\psi_k(C_k, \phi_k) = Max_{b_n}[\textstyle\int \{g_k(b_k|d_k)[b_k - d_k + \psi_{k+1}(C_k - a_k, \phi_k + \sigma_k^2)] +$$

$$[1 - g_k(b_k|d_k)]\psi_{k+1}(C_k, \phi_k)\}dF(d_j)].$$

As before, we have assumed that the $a_{ij} \sim N(\mu_{ij}, \sigma_{ij}^2)$ are independent in order to simplify our state defining variables.

It is important to note that $\Psi_k(\cdot, \cdot)$ is unconstrained here. Our chance-constrained formulation could have been similarly defined with

$$\begin{aligned}\psi_{n+1}(C_{n+1}, \phi_{n+1}) &= 0, && \text{if } \Pr(\textstyle\sum_{j \epsilon J_n} a_{ij} \leqq c_i) \geqq \alpha_i \forall i \in I,\\ &= -\infty, && \text{otherwise.}\end{aligned}$$

6. Delayed Disclosure of Contract Awardee Related to Level (γ)

(See Stark-Mayer [24].)

In earlier sections we have assumed that a contract award is announced immediately following the submission of bids and preceding the bid price deadline for the next contract. Thus one knows with certainty the outcome of the bid for contract j before being required to submit bid price b_{j+1}.

In many instances this is not the case; one must confront the inability to describe with certainty the amount of resources that must be committed to fulfill contracts *possibly* won at any moment during the bidding period. Here again the constructs of chance-constrained programming may be employed to bring the resource capacity constraints into play.

For example, consider the nth contract bid in the 1-period lag time case. With zero lag time this problem appears as:

$$\begin{aligned}\psi_n(C_n) &= \max_{b_n} g_n(b_n)[b_n - d_n], && \text{if } C_n - a_n \geqq 0,\\ &= 0, && \text{otherwise}\end{aligned}$$

[4] Introduced by Dantzig [7].

For the 1-peroid lag time problem we must re-cast our functional equation restrictions since C_n is a random variable assuming the value c_n with probability

$$(1 - g_{n-1}(b_{n-1})) \text{ and } c_n - a_{n-1}$$

with probability $g_{n-1}(b_{n-1})$. Thus we replace the restriction $C_n - a_n \geqq 0$ with the chance constraint $\Pr\{C_{n+1} \geqq 0\} \geqq \alpha$, for example, where the random variable C_{n+1} is defined by:

Sample value of C_{n+1}	$\Pr\{C_{n+1} = c_{n+1}\}$
c_n	$[1 - g_{n-1}(b_{n-1})][1 - g_n(b_n)]$,
$c_n - a_{n-1}$	$g_{n-1}(b_{n-1})[1 - g_n(b_n)]$,
$c_n - a_n$	$[1 - g_{n-1}(b_{n-1})]g_n(b_n)$,
$c_n - a_{n-1} - a_n$	$g_{n-1}(b_{n-1})g_n(b_n)$.

Hence, if $D_{n+1} = \{\text{sample values of } C_{n+1} \geqq 0\}$, we require that $\Pr\{D_{n+1}\} \geqq \alpha$ and can define our functional equation here as:

$$\begin{aligned}
\psi_k(C_k, b_{k-1}) &= \max_{b_k} \{g_k(b_k)[b_k - d_k] + g_{k-1}(b_{k-1})\psi_{k+1}(C_k - a_{k-1}, b_k) \\
&\quad + [1 - g_{k-1}(b_{k-1})]\psi_{k+1}(C_k, b_k)\}, \qquad \text{if } \Pr\{D_{k+1}\} \geqq \alpha \\
&= 0 + g_{k-1}(b_{k-1})\psi_{k+1}(C_k - a_{k-1}, \beta_k) \\
&\quad + [1 - g_{k-1}(b_{k-1})]\psi_{k+1}(C_k, \beta_k), \quad \text{otherwise}
\end{aligned}$$

for $k = 1, \cdots, n$ where $\psi_{n+1}(\cdot, \cdot) \equiv 0$.

It is easy to check the condition $\Pr\{D_{k+1}\} \geqq \alpha$ since $\Pr\{D_{k+1}\}$ is a nondecreasing function of b_{k+1}. However, if the time lag was for l periods, $l > 1$, then $\psi_k(\cdot)$ would require $l - 1$ more state parameters.

Appendix 2

THEOREM. *Assume in the linear programming model [I] of §3.3 that there is at least one profitable part, $p_j - d_j > 0$, which can be produced on machine 1, and at least one profitable part which can be produced on machine 2.*[5] *Then [I] has a basic optimal solution with at least $N - 2$ of the x_j variables 0 or k_j. (See [10], [13], [15], [21] for similar Theorems which stem from Manne's original dominance type theorems, [15].)*

PROOF. Since for each j, $0 \leqq x_j \leqq k_j$, it follows that an optimal solution to [I] exists. Constraints (1)–(5) are $N + 4$ in number and therefore at most $N + 4$ variables need be in an optimal basic solution. Because of the assumption, both S_1 and S_2 must be in the basis.

Now constraints (5) can be written $x_j + y_j = k_j$, where $y_j \geqq 0$ is a slack variable. Therefore since $k_j > 0$, it follows that either x_j or y_j are in an optimal solution for all j. Let p = number of x_j variables for which $x_j = 0$ or k_j. Then for $N - p$ variables both $x_j > 0$ and $y_j > 0$ occur, i.e., both are in the basis. Therefore the total number of variables that are in the basic optimal solution is

$$2 + p + 2(N - p),$$

[5] This assumption can be removed if we replace the right-hand sides of (3) and (4) in [I] by small negative numbers, say -0.0001.

and this number must be less than or equal to $N + 4$, i.e.,

$$2 + p + 2(N - p) \leqq N + 4$$

which implies

$$N - 2 \leqq p \qquad \text{Q.E.D.}$$

Note that the device of replacing 0 with, say, −0.0001, in both (3) and (4) will insure that both S_1 and S_2 are in any basic optimal solution.

References

1. Brummet, R. L., Overhead Costing, School of Business Administration, University of Michigan, Ann Arbor, 1957.
2. Byrne, R. F., "Chance-Constrained Programming and Related Approaches to Risk Control in Capital Budgeting," Management Sciences Research Report No. 131, Graduate School of Industrial Administration, Carnegie-Mellon University, June 1968.
3. ——, Charnes, A., Cooper, W. W., and Kortanek, K. O., "A Discrete Probability Chance-Constrained Capital Budgeting Model," (*Opsearch, The Journal of the Operations Research Society of India*), Vol. 6, No. 3, 4 Sept.–Dec., 1969.
4. Charnes, A. and W. W. Cooper, *Management Models and Industrial Applications of Linear Programming*, Vol. 2, Wiley, New York, 1961.
5. —— and Symonds, G. H., "Cost Horizons and Certainty Equivalents: An Approach to Stochastic Programming of Heating Oil," *Management Science*, Vol. 4, No. 3, April 1958, pp. 235–263.
6. —— and Stedry, A., "Investigations in the Theory of Multiple Budgeted Goals," Management Controls: New Directions in Basic Research, (Bonini, C. P., R. K. Jaedicke, and H. M. Wagner, Eds.), McGraw-Hill, New York, 1964.
7. Dantzig, G. B., "Linear Programming Under Uncertainty," *Management Science*, Vol. 1, No's. 3–4, 1955, pp. 197–206.
8. Horngren, Charles T., "A Contribution Margin Approach to the Analysis of Capacity Utilization," *The Accounting Review*, Vol. XLII, No. 1, April 1967, pp. 254–264.
9. Kaplan, Robert S. and Thompson, Gerald L., "Overhead Allocation via Mathematical Programming Models," *The Accounting Review*, Vol. XLVI, No. 2, April, 1971, 352–364.
10. Kortanek, K. O., Sodaro, D., and Soyster, A. L., "Multi-Product Production Scheduling via Extreme Point Properties of Linear Programming," *Naval Research Logistics Quarterly*, Vol. 15, No. 2, June, 1968, p. 287–300.
11. ——, Soden, J. V. and Sodaro, D., "On A Class of Sequential Probabilistic Bid Pricing Models," Technical Report No. 66, College of Engineering, Cornell University, Ithaca, New York, March, 1969.
12. —— and Soyster, A. L., "On the Status of Some Multi-Product Multi Period Production Scheduling Models," *Management Science*, Vol. 18, No. 8, April, 1971, B560-1.
13. Levitan, R. E., "A Note on Professor Manne's Dominance Theorem," *Management Science*, Vol. 5, No. 3, April, 1959.
14. Littlechild, Stephen C., "Marginal Cost Pricing with Joint Costs," *The Economic Review*, LXXX, 323–335, 1970.
15. Manne, A. S., "Programming of Economic Lot Sizes," *Management Science*, Vol. 4, No. 2, Jan., 1958.
16. Mobley, Sybil C., "Revenue Experience as a Guide to Asset Valuation," The Accounting Review, Vol. XLII, No. 1, Jan., 1967, pp. 114–123.
17. Samuels, J. M., "Opportunity Costing: An Application of Mathematical Programming," *Journal of Accounting Research*, Vol. 3, No. 2, Autumn 1965, pp. 182–191.
18. Shillinglaw, Gordon, *Cost Accounting: Analysis and Control*, Richard D. Irwin, Inc., Homewood, Illinois, 1961.
19. Shubik, Martin, *Strategy and Market Structure*, John Wiley, New York, 1959.
20. Soden, J. V. *Sequential Decision Making in Chance-Constrained Programming and Bid-Pricing Environments*, Ph.D. Thesis, Graduate School of Cornell University, September, 1969.
21. Soyster, A. L. *Multi-Product, Multi-Period Production Scheduling via Extreme Point Properties of Linear Programming and Some Horizon Posture Properties Over Time*, Masters Thesis, Graduate School of Cornell University, June, 1967.

22. Stark, Robert M., "Discussion for 'Bidding Strategies and Probabilities' by Gates (March 1967)," *Proc. Amer. Soc. Civil Eng., J. Construction Div.*, 110–112, January, 1968.
23. Stark, Robert M., "Competitive Bidding: A Comprehensive Bibliography," *Operations Research*, Vol. 19, No. 2, March–April, 1971, pp. 484–490.
24. Stark, Robert M. and Mayer, Robert H., Jr., "Some Multi-Contract Decision-Theoretic Competitive Bidding Models," *Operations Research*, Vol. 19, No. 2, March–April, 1971, 469–483.
25. Tucker, Spencer A., *Pricing for Higher Profit*, McGraw-Hill, New York, 1966.
26. ——, *Cost-Estimating and Pricing with Machine-Hour Rates*, Prentice-Hall, Englewood Cliffs, N. J.